Exam Analysis on Manifolds WBMA013-05

Tuesday 27.01.2026, 8:30 – 10:30

You are only allowed to use pen and paper during the exam, no additional material. Make sure to clearly explain the steps in your proofs and computations. The exam consists of two pages with a total of 4 exercises. You get 10 points for free.

DON'T PANIC
When a problem seems overwhelming, pause, breathe, and tackle it step by step.

Exercise 1. (6 + 8 + 10 = 24 points)

Let M and N be smooth manifolds and $F : M \to N$ a smooth function.

1. Explain in your own words what is a vector on a manifold and what is the tangent space.
2. Define the differential dF_p as a map between tangent spaces and explain how it helps to characterize the intutive idea that tangent spaces are the hyperplanes tangent to the manifold at the specific point.
3. Let now $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$ with the standard Euclidean atlas. For $x \in M$ and $v \in T_x M$, show that
$$dF_x(v) = \frac{d}{dt} F(x + tv)\Big|_{t=0}$$
with x, v interpreted as vectors in $\mathbb{R}^m$.

Exercise 2. (10 + 8 + 6 = 24 points)

Let $M_n(\mathbb{R})$ be the space of $n \times n$ matrices identified with $\mathbb{R}^{n^2}$. Let

$$O(n) = \{A \in M_n(\mathbb{R}) \mid A^T A = I_n\} \quad \text{and} \quad Sym(n) = \{S \in M_n(\mathbb{R}) \mid S^T = S\}$$

be the spaces of orthogonal and symmetric matrices respectively.

1. Show that $O(n)$ is a regular level set of the map
$$F : M_n(\mathbb{R}) \to Sym(n), \qquad F(A) := A^T A.$$
What is the dimension of $O(n)$?
Hint: for $O \in O(n)$ and $S \in Sym(n)$, what is $(OS)^T$?
Also, don't forget Exercise 1.3 (it can be used here without having solved the exercise)
2. Show that the tangent space $T_I O(n)$ at the identity matrix I is the space of skew-symmetric matrices, that is, matrices $Q \in M_n(\mathbb{R})$ such that $Q^T = -Q$.
3. Verify that the Lie bracket of two matrices in $T_I O(n)$ is also in $T_I O(n)$.

The exam continues on the back side.

Exercise 3. (8 + 8 + 8 = 24 points)

Let $M = \mathbb{R}^2$ with coordinates (x, y). Consider the two 1-forms:

$$\alpha = (1 + x^2)dy \quad \text{and} \quad \beta = xdy + ydx.$$

1. Compute the exterior derivatives $d\alpha$ and $d\beta$.
2. Find a function $f \in C^\infty(\mathbb{R}^2)$ such that $df = \beta$. Does such a function exist for α? Is any of the two forms closed? Justify your answer.
3. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the map $\Phi(u, v) = (u+v, uv)$. Compute the pullback $\Phi^*(\alpha \wedge \beta)$.

Exercise 4. (8 + 6 + 4 = 18 points)

1. Let M be an oriented n-manifold with boundary, ω a volume form on M and X a smooth outward-pointing vector field on ∂M. Show that $\iota_X \omega$ is a smooth nowhere-vanishing $(n-1)$-form on ∂M and thus, ∂M is orientable.
2. If $\{(U_i, \phi_i)\}$ is a positively oriented atlas on M with respect to ω, is it true that its restriction to ∂M is also positively oriented with respect to $\iota_X \omega$? Justify your answer.
3. State Stokes' theorem.

Exercise 1

1. Since manifolds are *not*, in general, vector spaces, a tangent vector at a point $p \in M$ of a smooth m-manifold M is *not*, in general, a vector in the usual sense (like in $\mathbb{R}^m$). At least not immediately.

A tangent vector on M is defined as a derivation on the algebra of smooth functions at p: it is a linear map $v : C^\infty(M) \to \mathbb{R}$ that satisfies the Leibniz rule:

$$v(fg) = f(p)v(g) + g(p)v(f).$$

The tangent space T_pM at $p \in M$ is the vector space of all tangent vectors at p.

It turns out that for a given chart (U, ϕ) around p, with coordinates (x^i), we can construct a basis of tangent vectors given by $\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}$ where for any $f \in C^\infty(M)$,

$$\frac{\partial}{\partial x^i}\bigg|_p (f) = \frac{\partial(f \circ \phi^{-1})}{\partial x^i}\bigg|_{\phi(p)}.$$

And thus T_pM is an m-dimensional vector space.

Note that in euclidean space, the derivations above are exactly the usual directional derivatives and thus a vector v in coordinates acts as the directional derivative along v. Moreover, using the basis just introduced, we can identify tangent vectors v with vectors in $\mathbb{R}^m$ via their components in this basis and see them also as vectors in the usual sense.

Borrowing again from the intuition from Euclidean spaces, one can show that tangent vectors can be also be interpreted as equivalence classes of curves passing through p with the same velocity v at p. This is convenient since if one can find the curve $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = p$ and $\gamma'(0) = v$, then for any $f \in C^\infty(M)$, one has

$$v(f) = \frac{d}{dt}(f \circ \gamma)\bigg|_{t=0}.$$

This will be convenient in the next part.

2. The differential of F at $p \in M$ is a linear map

$$dF_p : T_pM \to T_{F(p)}N$$

defined as follows: for $v \in T_pM$, $dF_p(v)$ is the tangent vector at $F(p)$ such that for any $g \in C^\infty(N)$,

$$dF_p(v)(g) = v(g \circ F).$$

The differential dF_p captures the idea of pushing forward tangent vectors from M to N.

If $P := F^{-1}(q) \subset M$ is a submanifold defined as the preimage of a regular value $q \in N$, Proposition 3.2.16 states that for each $p \in P$, the tangent space T_pP is given by

$$T_pP = \ker dF_p,$$

where I am omitting the inclusion $T_pP \subset T_pM$. This means that the tangent space to the submanifold P at p consists of those tangent vectors in T_pM that are annihilated by dF_p.

If one thinks of dF_p as a linear approximation of F near p, then the kernel of dF_p represents the directions in which one can move within the submanifold P without leaving it, thus justifying the idea that tangent spaces are hyperplanes tangent to the manifold at that specific point.

Alternatively, if one thinks of dF_p as the Jacobian matrix of F at p (in local coordinates), then again the kernel corresponds to the directions in which the function F does not change, and so at the vectors tangent to the image of F at that point, which again aligns with the idea of being tangent to the level set defined by F.

3. Since $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$ with the standard Euclidean atlas, we can identify tangent spaces with the spaces themselves:

$$T_x M \cong \mathbb{R}^m, \quad T_{F(x)} N \cong \mathbb{R}^n.$$

Let $v \in T_x M \cong \mathbb{R}^m$. We want to show that

$$dF_x(v) = \frac{d}{dt} F(x + tv)\Big|_{t=0}.$$

Method 1: direct computation. By definition of the differential, for any $g \in C^\infty(N)$,

$$dF_x(v)(g) = v(g \circ F).$$

Since v is a vector in $\mathbb{R}^m$ as a smooth manifold, it as a directional derivative:

$$v(g \circ F) = v^i \cdot \frac{\partial}{\partial x^i}(g \circ F)(x).$$

Using the chain rule, we have

$$\frac{d}{dt}(g \circ F)(x + tv)\Big|_{t=0} = \frac{\partial (g \circ F)}{\partial x^i}(x) \cdot \frac{d}{dt}(x^i + tv^i)\Big|_{t=0} = v^i \frac{\partial (g \circ F)}{\partial x^i}(x).$$

Method 2: using curves. Observe that $\gamma(t) = x + tv$ is a curve in M with $\gamma(0) = x$ and $\gamma'(0) = v$. Then, by the definition of the differential,

$$dF_x(v)(g) = v(g \circ F) = \frac{d}{dt}(g \circ F \circ \gamma)(t)\Big|_{t=0} = \frac{d}{dt}(g \circ F)(x + tv)\Big|_{t=0}.$$

In both cases, since this holds for all $g \in C^\infty(N)$, we conclude that

$$dF_x(v) = \frac{d}{dt} F(x + tv)\Big|_{t=0}.$$

Exercise 2

1. Since the map $F(A) = A^TA$ is a polynomial in the entries of A, it is smooth.

By definition,

$$O(n) = \{A \in M_n(\mathbb{R}) \mid A^TA = I_n\} = F^{-1}(I_n).$$

We need to show that I_n is a regular value of F, i.e., that for all $A \in O(n)$, the differential $dF_A : T_AM_n(\mathbb{R}) \to T_{F(A)}Sym(n)$ is surjective. Let $A \in O(n)$. For $H \in T_AM_n(\mathbb{R}) \simeq M_n(\mathbb{R})$, using 1.3, we have

$$dF_A(H) = \frac{d}{dt}F(A + tH)\Big|_{t=0} = A^TH + H^TA.$$

We want to show that for any $S \in Sym(n)$, there exists H such that $A^TH + H^TA = S$.

Let $H = \frac{1}{2}AS$. Then,

$$A^TH + H^TA = A^T\left(\frac{1}{2}AS\right) + \left(\frac{1}{2}AS\right)^T A = \frac{1}{2}S + \frac{1}{2}S^T = S.$$

This shows that $O(n)$ is a regular level set of F, so it is a smooth submanifold of $M_n(\mathbb{R})$.

To find the dimension of $O(n)$, we use the fact that

$$\dim M_n(\mathbb{R}) = n^2, \quad \dim Sym(n) = \frac{n(n+1)}{2}.$$

The latter comes from a countin argument: a symmetric matrix is determined by the n diagonal entries and the $\frac{n(n-1)}{2}$ entries above the diagonal.

By the regular levelset theorem,

$$\dim O(n) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

2. Since $O(n) = F^{-1}(I_n)$, the tangent space at I is the kernel of dF_I. For $H \in M_n(\mathbb{R})$,

$$dF_I(H) = I^TH + H^TI = H + H^T.$$

So,

$$T_IO(n) = \{H \mid H + H^T = 0\} = \{H \mid H^T = -H\},$$

the space of skew-symmetric matrices.

3. Let $Q_1, Q_2 \in T_IO(n)$, so $Q_1^T = -Q_1$, $Q_2^T = -Q_2$. Their Lie bracket is

$$[Q_1, Q_2] = Q_1Q_2 - Q_2Q_1.$$

One can check it by direct computation:

$$[Q_1, Q_2]^T = (Q_1Q_2)^T - (Q_2Q_1)^T = Q_2^TQ_1^T - Q_1^TQ_2^T = (-Q_2)(-Q_1) - (-Q_1)(-Q_2) = Q_2Q_1 -$$

So, $[Q_1, Q_2]$ is skew-symmetric, hence in $T_IO(n)$.

Exercise 3

1.

$$d\alpha = d(1+x^2)\wedge dy = 2xdx\wedge dy.$$
$$d\beta = d(xdy+ydx) = dx\wedge dy + dy\wedge dx = dx\wedge dy - dx\wedge dy = 0.$$

2. From the previous point, β is closed ($d\beta = 0$) and α is not closed.

To check if they are exact we can compare their differentials to df for some function f, identifying the basis elements.

For β we have

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = ydx + xdy.$$

So,

$$\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x.$$

Integrating either one, and replacing in the other, one finds

$$f(x,y) = xy + C,$$

where C is a constant.

For α, we can observe that if there exists f such that $df = \alpha$, then $d\alpha = d(df) = 0$. But we computed before that α is not closed:

$$d\alpha = 2xdx\wedge dy \neq 0,$$

and thus there can be no function f such that $df = \alpha$.

Note that we can also see this directly by trying to find f:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = (1+x^2)dy \quad \implies \quad \frac{\partial f}{\partial x} = 0, \quad \frac{\partial f}{\partial y} = 1+x^2.$$

The first equation implies that f does not depend on x, while the second one implies that f must depend on x, which is a contradiction.

3. Let fix some notation. From

$$\Phi(u,v) = (u+v, uv),$$

we have

$$x = u+v, \quad y = uv.$$

Thus,

$$dx = du + dv, \quad dy = vdu + udv.$$

We can also directly compute

$$\alpha\wedge\beta = (1+x^2)dy\wedge(xdy+ydx) = (1+x^2)ydy\wedge dx = -(1+x^2)ydx\wedge dy.$$

To compute the pullback we observe that pullbacks distribute over wedge products and commute with the exterior derivative, so:

$$\begin{aligned}\Phi^*(\alpha \wedge \beta) &= -(1+(u+v)^2)(uv)(d\Phi^*x) \wedge (d\Phi^*y) \\ &= -uv(1+(u+v)^2)(du+dv) \wedge (vdu+udv) \\ &= -uv(1+(u+v)^2)(u-v)du \wedge dv.\end{aligned}$$

Exercise 4

1. This is the proof of Proposition 10.2.15 in the lecture notes.

Since ω and X are smooth, the interior product $\eta = \iota_X \omega$ is also smooth.

We need to show that it is nowhere vanishing. Let $p \in \partial M$ be an arbitrary point on the boundary. We need to show that $\eta_p \neq 0$. Let $\{v_1, \dots, v_{n-1}\}$ be a basis for the tangent space of the boundary, $T_p(\partial M)$.

By definition of the interior product,

$$(\iota_X \omega)_p(v_1, \dots, v_{n-1}) = \omega_p(X_p, v_1, \dots, v_{n-1}).$$

Since X is outward-pointing, the vector X_p is transverse to the boundary; that is, $X_p \notin T_p(\partial M)$. Consequently, the set of vectors $\{X_p, v_1, \dots, v_{n-1}\}$ forms a linearly independent set (a basis) for the tangent space of the manifold, $T_p M$.

Since ω is a volume form on M, it is non-degenerate. Therefore, its evaluation on any basis of $T_p M$ is non-zero:

$$\omega_p(X_p, v_1, \dots, v_{n-1}) \neq 0.$$

Thus, $\iota_X \omega$ is non-zero on any basis of $T_p(\partial M)$, meaning it is nowhere-vanishing.

Since there exists a smooth, nowhere-vanishing $(n-1)$-form $\iota_X \omega$ on ∂M, the boundary ∂M is orientable.

2. No, it is not generally true. It fails when the dimension is odd. This is the remark after Proposition 10.2.15.

Whether the restriction of a positively oriented atlas on M yields a positively oriented atlas on ∂M (with respect to $\iota_X \omega$) depends on the dimension m and the convention used for the half-space model.

Consider the standard half-space model $\mathcal{H}^m = \{(x^1, \dots, x^m) \in \mathbb{R}^m \mid x^m \geq 0\}$. Let $(U, \phi) = (x^1, \dots, x^m)$ be a chart in the positively oriented atlas of M. In these local coordinates, the volume form is $\omega = f\, dx^1 \wedge \cdots \wedge dx^m$ for some positive smooth function f.

At the boundary ($x^m = 0$), the outward-pointing vector field is $X = -\frac{\partial}{\partial x^m}$. The induced form $\iota_X \omega$ is given by

$$\iota_{-\partial_m}(dx^1 \wedge \cdots \wedge dx^m) = -\iota_{\partial_m}(dx^1 \wedge \cdots \wedge dx^m).$$

Using the property $\iota_v(\alpha \wedge \beta) = (\iota_v \alpha) \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge (\iota_v \beta)$, as we did in class, we get

$$\iota_{-\partial_m}(dx^1 \wedge \cdots \wedge dx^m) = -(-1)^{m-1}(dx^1 \wedge \cdots \wedge dx^{m-1}) = (-1)^m (dx^1 \wedge \cdots \wedge dx^{m-1}).$$

The restriction of the chart to the boundary gives coordinates $(x^1, \dots, x^{m-1})$. The volume form associated with these coordinates is $dx^1 \wedge \cdots \wedge dx^{m-1}$.

Comparing the two forms on the boundary, we have:

- If m is even, $(-1)^m = 1$. The signs match, and the restriction is positively oriented.
- If m is odd, $(-1)^m = -1$. The signs are opposite, and the restriction is *negatively* oriented.

3. Stokes' theorem states the following:

Let M be an oriented, smooth n-manifold with boundary ∂M, and let ∂M be given the induced boundary orientation (defined by the outward-pointing normal vector).

If η is a smooth, compactly supported $(n-1)$-form on M, then:

$$\int_M d\eta = \int_{\partial M} \eta.$$

Here the integral on the right has to be intended as follows. Let $i : \partial M \hookrightarrow M$ be the inclusion map, then

$$\int_{\partial M} \eta := \int_{\partial M} i^* \eta.$$